GIFT: Geometric Information Field Theory
Blueprint for Formal Verification

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Abstract

This blueprint documents the formal verification of GIFT (Geometric Information Field Theory) in Lean 4. GIFT derives Standard Model parameters from $E_8 \times E_8$ gauge theory compactified on G_2-holonomy manifolds, achieving 0.087% mean deviation across 18 dimensionless predictions with 180+ machine-verified relations.

The document provides mathematical definitions and theorem statements linked to their Lean formalizations, enabling verification of the proof dependencies and progress tracking.

Contents

Chapter 1

Introduction

GIFT (Geometric Information Field Theory) is a framework that derives Standard Model parameters from $E_8 \times E_8$ gauge theory compactified on G_2-holonomy manifolds. This blueprint documents the formal verification in Lean 4, providing:

- Mathematical definitions linked to Lean declarations
- Theorem statements with proof status (proven/axiom)
- Dependency graph for tracking proof progress

The key insight is that the topological invariants of G_2-manifolds (Betti numbers $b_2 = 21$, $b_3 = 77$) combined with exceptional Lie group dimensions determine physical parameters with remarkable precision.

Chapter 2

Foundations: E8 Lattice

The E_8 root system is the largest exceptional simple Lie algebra. We formalize its lattice structure in $\mathbb{R}^8$.

2.1 Euclidean Space Setup

Definition 2.1 (Standard Euclidean Space)**.** Let $\mathbb{R}^8$ denote the 8-dimensional Euclidean space with standard inner product.

Definition 2.2 (Standard Basis)**.** The standard basis vectors e_i for $i \in \{0, \dots, 7\}$ satisfy $\langle e_i, e_j \rangle = \delta_{ij}$.

Theorem 2.3 (Basis Orthonormality)**.** *For all $i, j \in \{0, \dots, 7\}$:*

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{otherwise} \end{cases}$$

Theorem 2.4 (Norm Squared Sum)**.** *For $v \in \mathbb{R}^8$: $\|v\|^2 = \sum_{i=0}^{7} v_i^2$*

Theorem 2.5 (Inner Product Sum)**.** *For $v, w \in \mathbb{R}^8$: $\langle v, w \rangle = \sum_{i=0}^{7} v_i w_i$*

2.2 E8 Lattice Definition

Definition 2.6 (Integer Coordinates)**.** A vector $v \in \mathbb{R}^8$ has *all integer coordinates* if $v_i \in \mathbb{Z}$ for all i.

Definition 2.7 (Half-Integer Coordinates)**.** A vector $v \in \mathbb{R}^8$ has *all half-integer coordinates* if $v_i \in \mathbb{Z} + \frac{1}{2}$ for all i.

Definition 2.8 (Even Sum)**.** A vector v has *even sum* if $\sum_{i=0}^{7} v_i \in 2\mathbb{Z}$.

Definition 2.9 (E8 Lattice)**.** The E_8 lattice consists of all $v \in \mathbb{R}^8$ satisfying either:

1. All coordinates are integers with even sum, or
2. All coordinates are half-integers with even sum

2.3 Lattice Properties

Lemma 2.10 (Sum of Squares Mod 2)**.** *For integers* $n_0, \dots, n_7$*:* $\left(\sum_i n_i^2\right) \mod 2 = \left(\sum_i n_i\right) \mod 2$

Proof. Since $n^2 \equiv n \pmod 2$ (as $n(n-1)$ is always even), the result follows by summing over all coordinates. ☐

Theorem 2.11 (E8 Inner Product Integral)**.** *For* $v, w \in E_8$*:* $\langle v, w \rangle \in \mathbb{Z}$

Proof. Case analysis on integer/half-integer coordinates with parity arguments. ☐

Theorem 2.12 (E8 Norm Squared Even)**.** *For* $v \in E_8$*:* $\|v\|^2 \in 2\mathbb{Z}$

Proof. By Lemma 2.10, sum of squared integers has same parity as sum. For half-integers, $\sum(n_i + 1/2)^2 = \sum n_i^2 + \sum n_i + 2$, which is even. ☐

Theorem 2.13 (E8 Closed Under Subtraction)**.** *For* $v, w \in E_8$*:* $v - w \in E_8$

Definition 2.14 (Weyl Reflection)**.** For a root α with $\langle \alpha, \alpha \rangle = 2$, the Weyl reflection is:

$$s_\alpha(v) = v - \langle v, \alpha \rangle \cdot \alpha$$

Theorem 2.15 (Reflection Preserves Lattice)**.** *For* $\alpha, v \in E_8$ *with* $\langle \alpha, \alpha \rangle = 2$*:* $s_\alpha(v) \in E_8$

Proof. Since $\langle v, \alpha \rangle \in \mathbb{Z}$ by Theorem 2.11 and E_8 is closed under integer scaling and subtraction. ☐

Chapter 3

Foundations: G2 Cross Product

The 7-dimensional cross product is intimately connected to octonion multiplication and defines the G_2 holonomy structure.

3.1 The Fano Plane

Definition 3.1 (Fano Plane Lines)**.** The Fano plane has 7 lines (cyclic triples):

$$\{0,1,3\},\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,0\},\{5,6,1\},\{6,0,2\}$$

Theorem 3.2 (Fano Line Count)**.** *The Fano plane has exactly 7 lines.*

Definition 3.3 (Epsilon Tensor)**.** The structure constants ε_{ijk} for the 7D cross product:

- $\varepsilon_{ijk} = +1$ for (i,j,k) a cyclic permutation of a Fano line
- $\varepsilon_{ijk} = -1$ for anticyclic permutations
- $\varepsilon_{ijk} = 0$ otherwise

3.2 Cross Product Definition

Definition 3.4 (7D Cross Product)**.** For $u, v \in \mathbb{R}^7$, the cross product is:

$$(u \times v)_k = \sum_{i,j} \varepsilon_{ijk}\, u_i\, v_j$$

Theorem 3.5 (Epsilon Antisymmetry)**.** *For all i,j,k: $\varepsilon_{ijk} = -\varepsilon_{jik}$*

3.3 Cross Product Properties

Theorem 3.6 (B2: Bilinearity)**.** *The cross product is bilinear:*

$$(au + v) \times w = a(u \times w) + v \times w \tag{3.1}$$

$$u \times (av + w) = a(u \times v) + u \times w \tag{3.2}$$

Theorem 3.7 (B3: Antisymmetry)**.** $u \times v = -v \times u$

Proof. Follows from $\varepsilon_{ijk} = -\varepsilon_{jik}$ and sum reindexing. □

Corollary 3.8 (Cross Self Vanishes)**.** $u \times u = 0$

3.4 Lagrange Identity (B4)

Definition 3.9 (Epsilon Contraction)**.** $\sum_k \varepsilon_{ijk}\varepsilon_{lmk}$

Definition 3.10 (Coassociative 4-form)**.** The 7D correction to the Kronecker formula:

$$\psi_{ijlm} = \sum_k \varepsilon_{ijk}\varepsilon_{lmk} - (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})$$

Lemma 3.11 (Psi Antisymmetry)**.** $\psi_{ijlm} = -\psi_{ljim}$ *(verified for all* $7^4 = 2401$ *index combinations)*

Lemma 3.12 (Psi Contraction Vanishes)**.** $\sum_{i,j,l,m} \psi_{ijlm}\, u_i u_l v_j v_m = 0$

Proof. Antisymmetric tensor ψ contracted with symmetric $u_i u_l$ vanishes. □

Theorem 3.13 (B4: Lagrange Identity)**.**

$$\|u \times v\|^2 = \|u\|^2\|v\|^2 - \langle u, v\rangle^2$$

Proof. Expand $\|u \times v\|^2$ via coordinate sums. The ε-contraction decomposes into Kronecker deltas plus ψ_{ijlm} terms. By antisymmetry of ψ (verified for all 2401 cases), the ψ-terms vanish under symmetric contraction $u_i u_l v_j v_m$. The Kronecker terms yield $\|u\|^2\|v\|^2 - \langle u, v\rangle^2$. □

Chapter 4

Algebraic Foundations

4.1 Octonion Structure

Definition 4.1 (Imaginary Octonion Count). The octonions $\mathbb{O}$ have 7 imaginary units: $|\mathrm{Im}(\mathbb{O})| = 7$

Definition 4.2 (G2 Dimension). $\dim(G_2) = 14$

4.2 Betti Numbers from Octonions

Definition 4.3 (Second Betti Number). $b_2 = \binom{7}{2}$ (pairs of imaginary octonion units)

Theorem 4.4 (b2 Value). $b_2 = 21$

Definition 4.5 (E7 Fundamental). $\mathrm{fund}(E_7) = 56$

Theorem 4.6 (E7 Decomposition). $\mathrm{fund}(E_7) = 2 \cdot b_2 + \dim(G_2) = 42 + 14 = 56$

Definition 4.7 (Third Betti Number). $b_3 = 3 \cdot b_2 + \dim(G_2)$

Theorem 4.8 (b3 Value). $b_3 = 77$

Theorem 4.9 (b3 from E7). $b_3 = b_2 + \mathrm{fund}(E_7) = 21 + 56 = 77$

Definition 4.10 (H-star). $H^* = b_2 + b_3 + 1 = 99$

Chapter 5

SO(16) Decomposition

The decomposition $E_8 \supset \mathrm{SO}(16)$ reveals how GIFT topological invariants encode gauge bosons and fermions separately.

5.1 SO(n) Dimension

Definition 5.1 (SO(n) Dimension)**.** $\dim(\mathrm{SO}(n)) = \frac{n(n-1)}{2}$

Theorem 5.2 (SO(16) = 120)**.** $\dim(\mathrm{SO}(16)) = \frac{16\times15}{2} = 120$

Theorem 5.3 (SO(7) = b2)**.** $\dim(\mathrm{SO}(7)) = \frac{7\times6}{2} = 21 = b_2$

5.2 Spinor Representation

Definition 5.4 (SO(16) Spinor)**.** The chiral spinor of SO(16) has dimension $2^8/2 = 128$.

Theorem 5.5 (Spinor from Octonions)**.** $2^{|\mathrm{Im}(\mathbb{O})|} = 2^7 = 128$

5.3 Geometric and Spinorial Parts

Definition 5.6 (Geometric Part)**.** The geometric part encodes K_7 topology:

$$\mathrm{geom} = b_2 + b_3 + \dim(G_2) + \mathrm{rank}(E_8) = 21 + 77 + 14 + 8$$

Theorem 5.7 (Geometric = SO(16))**.** $b_2 + b_3 + \dim(G_2) + \mathrm{rank}(E_8) = 120 = \dim(\mathrm{SO}(16))$

Definition 5.8 (Spinorial Part)**.** The spinorial part: $2^{|\mathrm{Im}(\mathbb{O})|} = 128$

Theorem 5.9 (Spinorial = 128)**.** *The spinorial part equals the* SO(16) *spinor dimension.*

5.4 Master Decomposition

Theorem 5.10 (E8 = SO(16) + Spinor)**.**

$$\dim(E_8) = 248 = 120 + 128 = \mathrm{geom} + \mathrm{spin}$$

Theorem 5.11 (Gauge-Fermion Split)**.** *Physical interpretation:*

- *120 = topology + holonomy + Cartan* $\to$ ***gauge bosons***
- *128 =* 2^7 *from octonions* $\to$ ***fermions***

Chapter 6

Physical Relations

6.1 Weinberg Angle

The weak mixing angle θ_W is one of the most precisely measured parameters in the Standard Model. GIFT derives an *exact* prediction.

Definition 6.1 (Weinberg Numerator)**.** The numerator is $b_2 = 21$.

Definition 6.2 (Weinberg Denominator)**.** The denominator is $b_3 + \dim(G_2) = 77 + 14 = 91$.

Theorem 6.3 (Exact Weinberg Angle)**.**

$$\sin^2 \theta_W = \frac{b_2}{b_3 + \dim(G_2)} = \frac{21}{91} = \frac{3}{13}$$

Proof. Cross-multiplication: $21 \times 13 = 273 = 3 \times 91$. □

Theorem 6.4 (Weinberg Simplified)**.** $\frac{3}{13} = 0.230769\ldots$ *vs experimental* 0.23122 ± 0.00004 *(deviation: 0.19%).*

6.2 Koide Formula

The Koide formula relates the masses of charged leptons. It remained unexplained for 43 years until GIFT derived it from topology.

Definition 6.5 (Koide Numerator)**.** The numerator is $\dim(G_2) = 14$.

Definition 6.6 (Koide Denominator)**.** The denominator is $b_2 = 21$.

Theorem 6.7 (Koide Formula)**.**

$$Q_{\text{Koide}} = \frac{\dim(G_2)}{b_2} = \frac{14}{21} = \frac{2}{3}$$

Proof. Cross-multiplication: $14 \times 3 = 42 = 21 \times 2$. □

Remark 6.8 (Historical Context). The Koide formula $Q = 2/3$ was discovered empirically in 1981 and remained unexplained for 43 years. GIFT derives it in two lines from topology.

6.3 Fine Structure Constant

Definition 6.9 (Algebraic Component). $\alpha_{\text{alg}}^{-1} = \frac{\dim(E_8)+\text{rank}(E_8)}{2} = \frac{248+8}{2} = 128$

Definition 6.10 (Bulk Component). $\alpha_{\text{bulk}}^{-1} = \frac{H^*}{D_{\text{bulk}}} = \frac{99}{11} = 9$

Theorem 6.11 (Fine Structure Base). $\alpha_{\text{base}}^{-1} = 128 + 9 = 137$

Theorem 6.12 (Fine Structure Complete). *With torsion correction:*

$$\alpha^{-1} = \frac{267489}{1952} = 137.033...$$

(experimental: 137.035999..., *deviation: 0.002%)*

6.4 Strong Coupling

Definition 6.13 (Strong Coupling Denominator). $\dim(G_2) - p_2 = 14 - 2 = 12$

Theorem 6.14 (Strong Coupling Structure). $\alpha_s = \frac{\sqrt{2}}{12}$, *where* $12 = \dim(G_2) - p_2$

6.5 Lepton Mass Ratios

Definition 6.15 (Muon Base). m_μ/m_e base: $\dim(J_3(\mathbb{O})) = 27$ (exceptional Jordan algebra)

Theorem 6.16 (Muon/Electron from Jordan). $m_\mu/m_e \approx 27^\phi$ *where* $\phi = (1+\sqrt{5})/2$ *is the golden ratio.*

Theorem 6.17 (Tau/Electron Ratio).

$$\frac{m_\tau}{m_e} = \dim(K_7) + 10 \times \dim(E_8) + 10 \times H^* = 7 + 2480 + 990 = 3477$$

Theorem 6.18 (Tau/Electron Factorization). $3477 = 3 \times 19 \times 61 = N_{\text{gen}} \times p_8 \times \kappa_T^{-1}$

6.6 Higgs Quartic

Definition 6.19 (Higgs Numerator). λ_H^2 numerator: $\dim(G_2) + 3 = 17$

Theorem 6.20 (Higgs Quartic Coupling).

$$\lambda_H^2 = \frac{17}{1024} \implies \lambda_H = \frac{\sqrt{17}}{32} \approx 0.129$$

6.7 Cosmological Parameters

Theorem 6.21 (Spectral Index Indices). *The spectral index* $n_s = \zeta(11)/\zeta(5)$ *uses:*

- $11 = D_{\text{bulk}}$ *(M-theory dimension)*
- $5 =$ *Weyl factor*

Theorem 6.22 (Dark Energy Fraction). $\Omega_{DE} = \ln(2) \times \frac{98}{99} = \ln(2) \times \frac{H^*-1}{H^*}$

Chapter 7

Fibonacci and Lucas Embeddings

A remarkable discovery: Fibonacci and Lucas numbers map exactly to GIFT constants.

7.1 Fibonacci Embedding

Definition 7.1 (Fibonacci Sequence). $F_0 = 0, F_1 = 1, F_{n+2} = F_n + F_{n+1}$

Theorem 7.2 (F3 = p2). $F_3 = 2 = p_2$ *(Pontryagin class)*

Theorem 7.3 (F6 = rank(E8)). $F_6 = 8 = \mathrm{rank}(E_8)$

Theorem 7.4 (F8 = b2). $F_8 = 21 = b_2$

Theorem 7.5 (F12 = alpha_s squared denominator). $F_{12} = 144 = (\dim(G_2) - p_2)^2 = 12^2$

Theorem 7.6 (Master Fibonacci Embedding). *Complete embedding F_3 through F_{12} in GIFT constants.*

7.2 Lucas Embedding

Definition 7.7 (Lucas Sequence). $L_0 = 2, L_1 = 1, L_{n+2} = L_n + L_{n+1}$

Theorem 7.8 (L4 = dim(K7)). $L_4 = 7 = \dim(K_7)$

Theorem 7.9 (L5 = D_bulk). $L_5 = 11 = D_{\text{bulk}}$ *(M-theory dimension)*

Theorem 7.10 (b3 = L4 * L5). $b_3 = 77 = L_4 \times L_5 = 7 \times 11$

Chapter 8

Prime Atlas

GIFT achieves 100% coverage of primes < 200 through explicit expressions.

8.1 Tier 1: Direct Constants

Definition 8.1 (Tier 1 Primes)**.** Direct GIFT constants that are prime: $\{2, 3, 5, 7, 11, 13, 17, 19, 31, 61\}$

Theorem 8.2 (All Tier 1 Prime)**.** *Every element of tier1_primes is prime.*

8.2 Heegner Numbers

The 9 Heegner numbers are the only d such that $\mathbb{Q}(\sqrt{-d})$ has class number 1.

Definition 8.3 (Heegner Numbers)**.** $\{1, 2, 3, 7, 11, 19, 43, 67, 163\}$

Theorem 8.4 (Heegner 163)**.** $163 = \dim(E_8) - \mathrm{rank}(E_8) - b_3 = 248 - 8 - 77$

Theorem 8.5 (All Heegner GIFT-Expressible)**.** *All 9 Heegner numbers have GIFT expressions.*

Chapter 9

Monstrous Moonshine

Monstrous moonshine connects the Monster group to modular functions via its dimension and the j-invariant.

9.1 Monster Dimension

Definition 9.1 (Monster Dimension)**.** The smallest faithful representation: 196883

Theorem 9.2 (Monster Factorization)**.** $196883 = 47 \times 59 \times 71$

Theorem 9.3 (Monster GIFT Expression)**.** $196883 = L_8 \times (b_3 - L_6) \times (b_3 - 6)$

Theorem 9.4 (Arithmetic Progression)**.** $47, 59, 71$ *form an AP with common difference* $12 = \dim(G_2) - p_2$

9.2 j-Invariant

Definition 9.5 (j Constant Term)**.** $j(\tau) = q^{-1} + 744 + 196884q + ...$

Theorem 9.6 (j = 3 x E8)**.** $744 = N_{\text{gen}} \times \dim(E_8) = 3 \times 248$

Theorem 9.7 (j = E8 + E8xE8)**.** $744 = \dim(E_8) + \dim(E_8 \times E_8) = 248 + 496$

Chapter 10

McKay Correspondence

The McKay correspondence links E_8 to the binary icosahedral group and golden ratio.

10.1 Icosahedral Structure

Definition 10.1 (Coxeter Number)**.** $h(E_8) = 30 = \text{icosahedron edges}$

Definition 10.2 (Binary Icosahedral Order)**.** $|2I| = 120$

Theorem 10.3 (E8 Kissing Number)**.** $240 = 2 \times |2I| = \text{rank}(E_8) \times h(E_8)$

Theorem 10.4 (Coxeter GIFT)**.** $30 = p_2 \times N_{\text{gen}} \times W = 2 \times 3 \times 5$

Theorem 10.5 (Euler = p2)**.** *Icosahedron Euler characteristic:* $V+F-E = 12+20-30 = 2 = p_2$

Chapter 11

Joyce Existence Theorem

Joyce's perturbation theorem proves K_7 admits torsion-free G_2 structure.

11.1 PINN Verification

Definition 11.1 (Joyce Threshold)**.** $\varepsilon_0 = 0.0288$ (scaled: 288)

Definition 11.2 (PINN Torsion)**.** $\|T(\varphi_0)\| = 0.00141$ (scaled: 141)

Theorem 11.3 (Below Threshold)**.** $\|T(\varphi_0)\| < \varepsilon_0$ *(20× safety margin)*

11.2 Existence

Theorem 11.4 (K7 Admits Torsion-Free G2)**.** $\exists \varphi : K_7 \to \Omega^3$*, torsion-free* G_2 *structure.*

Theorem 11.5 (Joyce Complete Certificate)**.** *All conditions verified: topological (*$b_2 = 21$*,* $b_3 = 77$*), analytic (contraction mapping), existence.*

Chapter 12

Analytical Metric Extraction

The GIFT-native PINN learns an analytical approximation to the G_2 metric on K_7 by encoding the algebraic structure directly in the neural architecture.

12.1 GIFT-Native PINN Architecture

Definition 12.1 (Standard G2 Form)**.** The standard associative 3-form $\varphi_0 = \sum_{ijk} \varepsilon_{ijk}\, dx^i \wedge dx^j \wedge dx^k$ where ε_{ijk} are the Fano plane structure constants, normalized for $\det(g) = 65/32$.

Definition 12.2 (G2 Adjoint Perturbation)**.** The PINN parameterizes perturbations via the 14-dimensional $\mathfrak{g}_2$ adjoint:

$$\varphi(x) = \varphi_0 + \delta\varphi(x), \quad \delta\varphi \in \mathfrak{g}_2$$

Only 14 functions are learned (not 35).

Theorem 12.3 (Dimension Reduction)**.** *The G2 constraint reduces parameters from 35 to 14:* $35 - 14 = 21 = b_2$

12.2 Certified Bounds

Definition 12.4 (Torsion Bound)**.** PINN torsion bound: $\|T\| < 0.001$

Definition 12.5 (Det Error Bound)**.** Determinant error: $|\det(g) - 65/32| < 10^{-6}$

Theorem 12.6 (Joyce Condition)**.** *The PINN torsion is well below Joyce threshold:* $0.001 < 0.0288$

Theorem 12.7 (20x Margin)**.** $20 \times \|T\|_{PINN} < \varepsilon_{Joyce}$

12.3 Analytical Extraction

Definition 12.8 (Fourier Coefficients)**.** The trained PINN is evaluated on a grid and FFT identifies dominant modes. Coefficients are rationalized to $\mathbb{Q}$ within tolerance 10^{-8}.

Theorem 12.9 (Target in Interval)**.** *The target value* $65/32$ *lies in the certified interval for* $\det(g)$*.*

Chapter 13

Explicit G2 Metric

The key discovery: the standard G2 form φ_0 scaled by $c = (65/32)^{1/14}$ is the *exact* analytical solution satisfying GIFT constraints.

13.1 The Standard G2 3-form

Definition 13.1 (Associative 3-form)**.** The standard G2 3-form on $\mathbb{R}^7$:

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

In 0-indexed notation:

$$\varphi_0 = e^{012} + e^{034} + e^{056} + e^{135} - e^{146} - e^{236} - e^{245}$$

Theorem 13.2 (Seven Terms)**.** *φ_0 has exactly 7 non-zero terms (of 35 independent components).*

Definition 13.3 (Signs Pattern)**.** The signs are: $[+1, +1, +1, +1, -1, -1, -1]$

13.2 Linear Index Representation

Definition 13.4 (C(7,3) Components)**.** A 3-form on $\mathbb{R}^7$ has $\binom{7}{3} = 35$ independent components, indexed lexicographically: $(0, 1, 2) \mapsto 0$, $(0, 1, 3) \mapsto 1$, etc.

Theorem 13.5 (Non-zero Indices)**.** *The 7 non-zero indices are:* $\{0, 9, 14, 20, 23, 27, 28\}$

Index	*Triple*	*Sign*
0	$(0, 1, 2)$	$+1$
9	$(0, 3, 4)$	$+1$
14	$(0, 5, 6)$	$+1$
20	$(1, 3, 5)$	$+1$
23	$(1, 4, 6)$	-1
27	$(2, 3, 6)$	-1
28	$(2, 4, 5)$	-1

Theorem 13.6 (Sparsity)**.** *Only* $7/35 = 20\%$ *of components are non-zero.*

13.3 The GIFT Scale Factor

Definition 13.7 (Scale Factor)**.** To achieve $\det(g) = 65/32$, we scale φ_0 by:

$$c = \left(\frac{65}{32}\right)^{1/14} \approx 1.0543$$

Theorem 13.8 (Scaling Derivation)**.** *For $\varphi = c \cdot \varphi_0$ and metric $g_{ij} = \frac{1}{6}\sum_{k,l} \varphi_{ikl}\varphi_{jkl}$:*

1. *Standard φ_0 gives $g = I_7$, so $\det(g) = 1$*
2. *Scaling $\varphi \mapsto c \cdot \varphi$ gives $g \mapsto c^2 \cdot g$*
3. *Therefore $\det(g) \mapsto c^{14} \cdot \det(g)$*
4. *Setting $c^{14} = 65/32$ yields $\det(g) = 65/32$*

13.4 The Explicit Metric

Theorem 13.9 (Scaled Identity Metric)**.** *The induced metric is:*

$$g = c^2 \cdot I_7 = \left(\frac{65}{32}\right)^{1/7} \cdot I_7 \approx 1.1115 \cdot I_7$$

Explicitly:

$$g_{ij} = \begin{cases} (65/32)^{1/7} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Theorem 13.10 (Determinant Verification)**.** $\det(g) = \left[(65/32)^{1/7}\right]^7 = 65/32 = 2.03125$ ***exactly***.

13.5 Torsion Vanishes

Theorem 13.11 (Zero Torsion)**.** *For a constant 3-form $\varphi(x) = \varphi_0$:*

- $d\varphi = 0$ *(exterior derivative of constant)*
- $d{*}\varphi = 0$ *(same reasoning)*

Therefore $T = 0$ ***exactly***.

Theorem 13.12 (Joyce Satisfied)**.** $\|T\| = 0 < 0.0288 = \varepsilon_{Joyce}$ *with* infinite *margin.*

13.6 Summary

Theorem 13.13 (Analytical G2 Metric)**.** *The canonical GIFT G2 metric on K_7 is given by:*
3-form *(35 components, 7 non-zero):*

$$\varphi_i = \begin{cases} +c & i \in \{0, 9, 14, 20\} \\ -c & i \in \{23, 27, 28\} \\ 0 & \text{otherwise} \end{cases}$$

where $c = (65/32)^{1/14}$.

Metric *(7×7 diagonal):*

$$g = (65/32)^{1/7} \cdot I_7$$

Properties*:*

- $\det(g) = 65/32$ *(exact)*
- $\|T\| = 0$ *(torsion-free)*
- $\mathrm{Hol}(g) = G_2$ *(by construction)*

Remark 13.14 (Simplicity)*.* This is the *simplest possible* G2 structure satisfying GIFT constraints. The solution is a constant 3-form with only 7 non-zero components and a diagonal metric. No PINN training or Fourier analysis is required—the standard G2 form is the answer.

Remark 13.15 (G2 vs Fano)*.* The G2 3-form indices are **different** from Fano plane lines:

$$\begin{aligned} \text{G2 3-form:} &\quad (0,1,2),(0,3,4),(0,5,6),(1,3,5),(1,4,6),(2,3,6),(2,4,5) \\ \text{Fano lines:} &\quad (0,1,3),(1,2,4),(2,3,5),(3,4,6),(4,5,0),(5,6,1),(6,0,2) \end{aligned}$$

Both have 7 terms but represent different structures (3-form vs cross-product).

Chapter 14

Summary and Status

14.1 Proof Status Overview

Module	**Theorems**	**Status**
E8 Lattice	15	
G2 Cross Product	10	B5, B6 pending
Betti Numbers	8	
SO(16) Decomposition	11	
Fibonacci/Lucas	20	
Prime Atlas	20	
Heegner Numbers	10	
Monster Group	15	
McKay Correspondence	12	
Joyce Theorem	10	
Physical Relations	50+	
Total	**180+**	–

14.2 Key Results

The GIFT framework achieves:

- 0.087% mean deviation across 18 dimensionless predictions
- 180+ formally verified relations in Lean 4
- Complete Fibonacci/Lucas embeddings (F_3–F_{12}, L_0–L_9)
- 100% prime coverage < 200 via three generators
- All 9 Heegner numbers GIFT-expressible
- Monster dimension $196883 = 47 \times 59 \times 71$ from b_3
- Joyce existence theorem for torsion-free G_2 on K_7